A and B want to divide a piece of cake into two equal parts; how do they do it so that each is satisfied he got at least a fair share? “Cake Cutting Algorithms: Be Fair If You Can” by Jack Robertson and William Webb is a book that answers questions such as this and discusses related but more complex problems.

The classic solution to the question posed above is called “cut and choose” – one person divides the cake into what he estimates are two equal parts and the other chooses a piece. It is straightforward to see why this works.

But what if there are more than two people? Or instead of equal halves, it is agreed to divide the cake in some other proportion, say a third to A and the rest to B?
These are the types of problems that the field of “Fair Division” seeks to answer. Incidentally, this branch of knowledge was started in the 1940’s by the famous Polish mathematicians Banach, Steinhaus and Knaster.

The problem of fair division for more than two players is more complicated than one might assume at first sight and the authors present several plausible looking schemes which fail. After pointing out the drawbacks of the schemes which fail, they also present several algorithms that will work. I will describe two of them here, which I found very elegant.

In the Moving Knife Algorithm, a knife is continuously passed over the cake from left to right and the first person who thinks the portion between the starting position and current position of the knife is 1/N, can shout “stop” and he gets that piece and drops out. This step is repeated on the remaining part of the cake and the remaining N-1 players. When only one person is left he takes what is left.

In the Trimming Algorithm, Player P1 cuts what he estimates to be a 1/N of the cake. This is passed to players P2, P3, Pn successively; any player who thinks he has been passed a piece bigger than 1/N trims it so that the reduced value is exactly 1/N in his estimation. When the piece reaches the last player, he can either accept it or else this piece is given to the last person who trimmed the piece. This person will drop out. The remaining N players will repeat the procedure with the remaining cake and as before, when only one person is left, he takes whatever part of the cake remains.

If two players are to divide the cake in non-equal parts, say 2/5 to A and 3/5 to B, it may be tempting to “clone 2 of A” and “clone 3 of B” and apply the above algorithms, but the authors point out that if the ratio is irrational, say A gets 1/square_root (2) and B gets the rest, then there are fundamentally different approaches needed.

The book also discusses fair division in settings under disagreement, i.e., scenarios where different players disagree about the value of different pieces of the cake, and the authors show that in these situations there is always a solution where there exists not only a solution where everyone believes he got a fair share but also, there will be some part of the cake left over. As a simple (and unrealistic) example, if A estimates the value of the two pieces as 40% and 60% whereas B estimates them as 60% and 40%, then giving A the second piece and B the first piece will result in each getting a piece he believes to be bigger.

A more realistic presented in the book shows the serendipity of disagreement. Suppose N persons have claims to equal shares of a piece of real estate which is bounded on the North buy a lake (curvilinear, irregular boundary), to the east and west by vertical boundaries and to the south by a horizontal road. They agree to have equal access to road and beach; how does division proceed so every one is satisfied? The authors suggest (and prove the correctness of) a really interesting algorithm. Each person is given an aerial picture of the real estate and told to divide it into N equal parts. If there is complete agreement, that is, when all the pictures are superimposed, the markings made by each player is identical to that of every other player), then it is trivial to see that a fair division is possible. But the more intriguing result is that if there is not complete agreement, it is then possible to

a) Give each player a division that he estimates to be a fair share and at the same time,
b) Still have some land left unassigned.

How this done employs is a fascinating algorithm and is described as well as proved in this book, which alone made it a worthwhile read for me.

As in any book of algorithms, there is analysis of the number of steps (or number of cuts) needed for cutting the cake, what is possible or impossible in a given number of cuts.

There are other interesting situations discussed – as an example, “envy free division” where every player gets by his estimate at least a fair share and also that no one got a bigger share than him. Also described is the “envy free dirty job division”, where N people have to do a dirty job, everyone feels he is allocated no more than 1/N of the dirty job and also that no one got a lesser share than him.

Pareto Optimality is discussed. There is an example where authors discuss how to make risk free bets – i.e., when there are two players who have different opinions on the possible result of a sport match, how it be possible for a third player to structure bets with both of them so as to assure himself of a riskless profit.